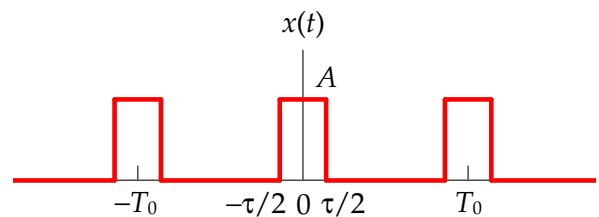
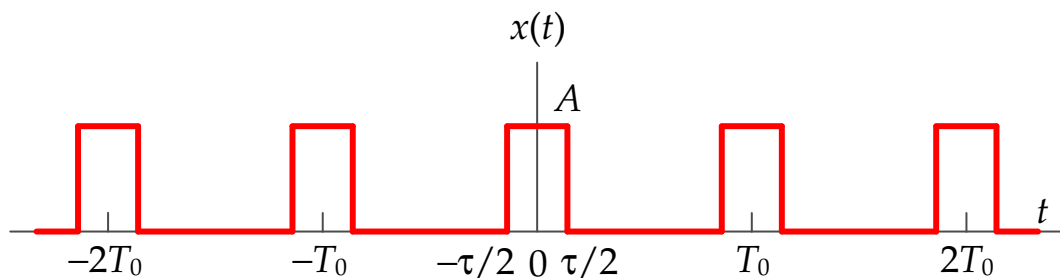


14. Fourier Series: Examples

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Q1. For the following signal $x(t) = \text{rep}_{T_0} \left\{ A \text{rect} \left(\frac{t}{\tau} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q1. Solution. The complex exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T_0}$$

where the coefficients are calculated as

$$\alpha_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\alpha_n = \frac{1}{T_0} \left[\int_{-T_0/2}^{-\tau/2} 0 \times (e^{-jn\omega_0 t}) dt + \int_{-\tau/2}^{\tau/2} A(e^{-jn\omega_0 t}) dt + \int_{\tau/2}^{T_0/2} 0 \times (e^{-jn\omega_0 t}) dt \right]$$

$$\alpha_n = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A(e^{-jn\omega_0 t}) dt$$

$$\alpha_0 = \frac{A\tau}{T_0}$$

$$\alpha_n = \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right), \quad n \neq 0$$

Hence,

$$\alpha_n = \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) = \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right), \quad n = 0, \pm 1, \pm 2, \dots$$

Notice that $\alpha_0 = \frac{A\tau}{T_0} \operatorname{sinc}(0) = \frac{A\tau}{T_0}$, which is average value or DC value.

The value of α_n is real, and hence, the phase is 0° or 180° .

$$|\alpha_n| = \left| \frac{A\tau}{T_0} \operatorname{sinc} \left(\frac{n\omega_0\tau}{2\pi} \right) \right|$$

$$\angle \alpha_n = 0^\circ \text{ or } 180^\circ$$

Notice that when α_n is negative, the phase is $180^\circ \equiv -180^\circ \equiv \pi \equiv -\pi$.

Since $x(t)$ is real-valued signal, we have

$$\alpha_{-n} = \alpha_n^* , \quad n = 0, 1, 2, 3, \dots$$

$$|\alpha_{-n}| = \left| \frac{A\tau}{T_0} \operatorname{sinc} \left(\frac{n\omega_0\tau}{2\pi} \right) \right| = |\alpha_n|$$

$$\angle \alpha_{-n} = 0^\circ \text{ or } 180^\circ = -\angle \alpha_n$$

The trigonometric Fourier series is

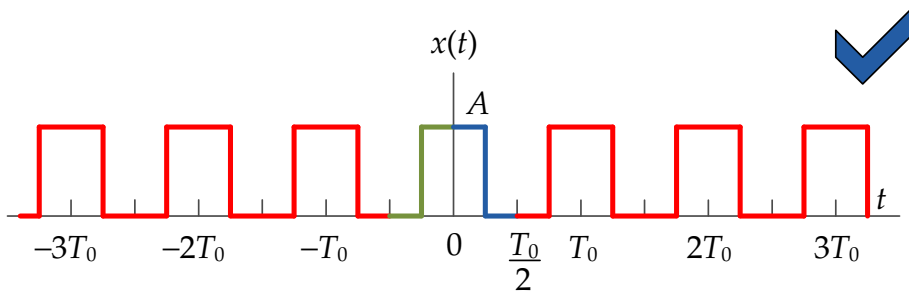
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] , \quad \omega_0 = \frac{2\pi}{T_0}$$

where the coefficients are calculated as

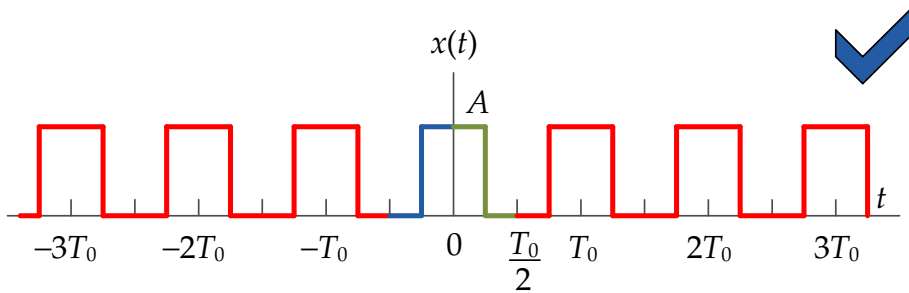
$$a_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(n\omega_0 t) dt , \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(n\omega_0 t) dt , \quad n = 0, 1, 2, 3, \dots$$

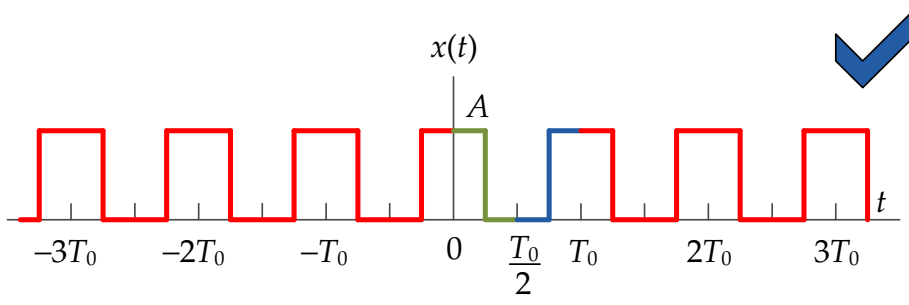
Since $x(t)$ is even, then $b_n = 0, \forall n$, and $a_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \cos(n\omega_0 t) dt$



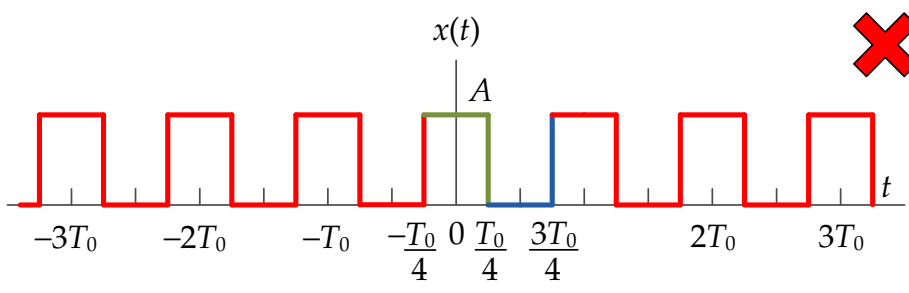
$$a_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$



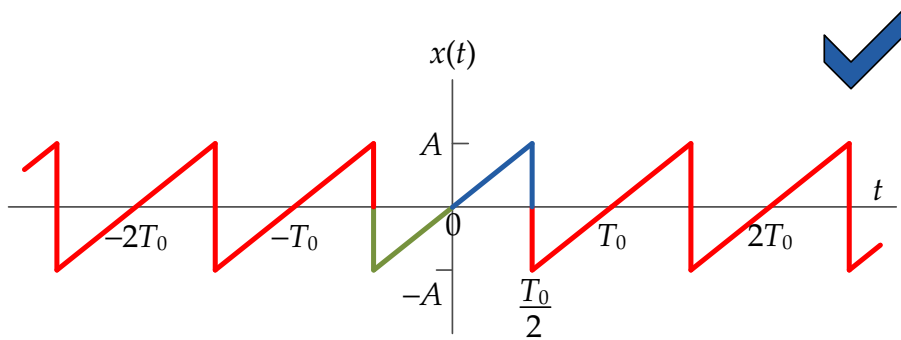
$$a_n = \frac{4}{T_0} \int_{-T_0/2}^0 x(t) \cos(n\omega_0 t) dt$$



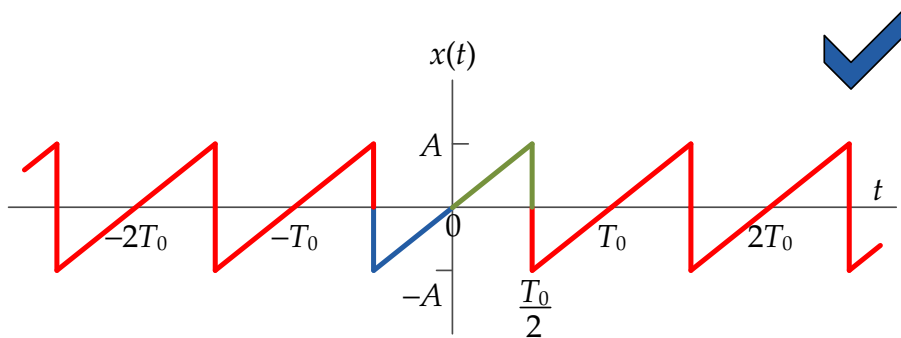
$$a_n = \frac{4}{T_0} \int_{T_0/2}^{T_0} x(t) \cos(n\omega_0 t) dt$$



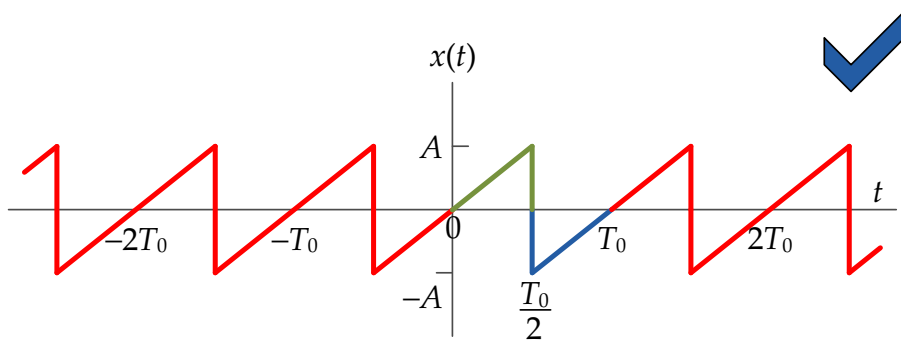
$$a_n \neq \frac{4}{T_0} \int_{T_0/4}^{3T_0/4} x(t) \cos(n\omega_0 t) dt$$



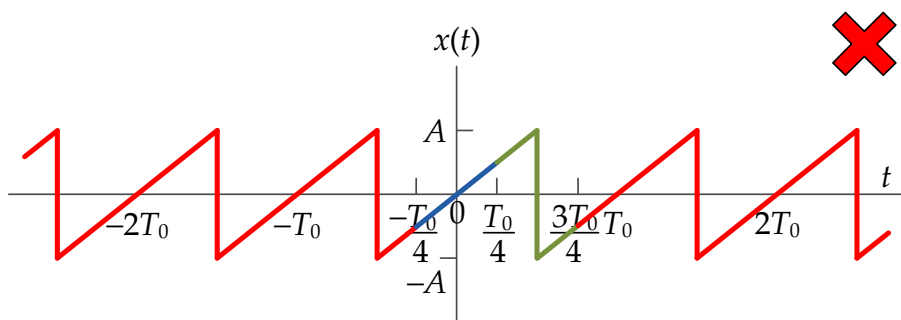
$$b_n = \frac{4}{T_0} \int_{T_0/2}^{T_0} x(t) \sin(n\omega_0 t) dt$$



$$b_n = \frac{4}{T_0} \int_{-T_0/2}^0 x(t) \sin(n\omega_0 t) dt$$



$$b_n = \frac{4}{T_0} \int_{T_0/2}^{T_0} x(t) \sin(n\omega_0 t) dt$$



$$b_n \neq \frac{4}{T_0} \int_{-T_0/4}^{T_0/4} x(t) \sin(n\omega_0 t) dt$$

$$a_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \cos(n\omega_0 t) dt = \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

$$a_n = \frac{4}{T_0} \int_0^{\tau/2} A \cos(n\omega_0 t) dt + \frac{4}{T_0} \int_{\tau/2}^{T_0/2} 0 \times \cos(n\omega_0 t) dt$$

$$a_n = \frac{4}{T_0} \int_0^{\tau/2} A \cos(n\omega_0 t) dt + 0$$

$$a_0 = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) dt = \frac{4}{T_0} \int_0^{\tau/2} A dt = \frac{4A}{T_0} [t]_0^{\tau/2} = \frac{4A}{T_0} \left[\frac{\tau}{2} - 0 \right] = \frac{2A\tau}{T_0}$$

Hence, the average value is $\frac{a_0}{2} = \frac{A\tau}{T_0}$, just like earlier.

$$a_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \cos(n\omega_0 t) dt = \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

$$a_n = \frac{4}{T_0} \int_0^{\tau/2} A \cos(n\omega_0 t) dt = \frac{4A}{T_0} \times \frac{1}{n\omega_0} \times [\sin(n\omega_0 t)]_0^{\tau/2}, \quad n \neq 0$$

$$a_n = \frac{4A}{T_0} \times \frac{1}{n\omega_0} \times \left[\sin\left(\frac{n\omega_0 \tau}{2}\right) - \sin(0) \right]$$

$$a_n = \frac{4A}{T_0} \times \frac{\sin\left(\frac{n\omega_0 \tau}{2}\right)}{n\omega_0} = \frac{2A\tau}{T_0} \times \frac{\sin\left(\frac{\pi \frac{n\omega_0 \tau}{2\pi}}{\pi \frac{n\omega_0 \tau}{2\pi}}\right)}{\pi \frac{n\omega_0 \tau}{2\pi}} = \frac{2A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0 \tau}{2\pi}\right)$$

Since $x(t)$ is an even signal, we have

$$b_n = 0$$

Notice that

$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) - j0, \quad n = 0, 1, 2, 3, \dots$$

Also,

$$\begin{aligned} a_n &= 2 \operatorname{Re}\{\alpha_n\}, & n &= 0, 1, 2, 3, \dots \\ b_n &= -2 \operatorname{Im}\{\alpha_n\}, & n &= 0, 1, 2, 3, \dots \end{aligned}$$

The compact Fourier series is

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n), \quad \omega_0 = \frac{2\pi}{T_0}$$

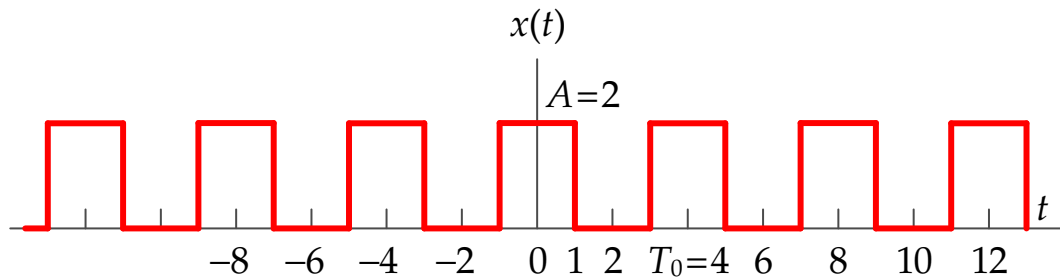
$$c_n = \sqrt{a_n^2 + b_n^2} = \sqrt{a_n^2 + 0^2} = |a_n| = \left| \frac{2A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) \right|$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{0}{a_n}\right) = 0^\circ \text{ or } 180^\circ$$

$$c_n = 2|\alpha_n|$$

$$\theta_n = -\angle\alpha_n = \tan^{-1}\left(\frac{-\operatorname{Im}\{\alpha_n\}}{\operatorname{Re}\{\alpha_n\}}\right)$$

Q2. For the following signal $x(t) = \text{rep}_4 \left\{ 2 \text{rect} \left(\frac{t}{2} \right) \right\}$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q2. Solution. You can do the integrals again, or just substitute to get

$$\alpha_n = \frac{A\tau}{T_0} \text{sinc} \left(\frac{n\omega_0\tau}{2\pi} \right) = \frac{A\tau}{T_0} \text{sinc} \left(\frac{n\tau}{T_0} \right) = \frac{2 \times 2}{4} \text{sinc} \left(\frac{n \times 2}{4} \right) = \text{sinc} \left(\frac{n}{2} \right)$$

$$a_n = \frac{2A\tau}{T_0} \text{sinc} \left(\frac{n\omega_0\tau}{2\pi} \right) = 2 \text{sinc} \left(\frac{n}{2} \right)$$

$$b_n = 0$$

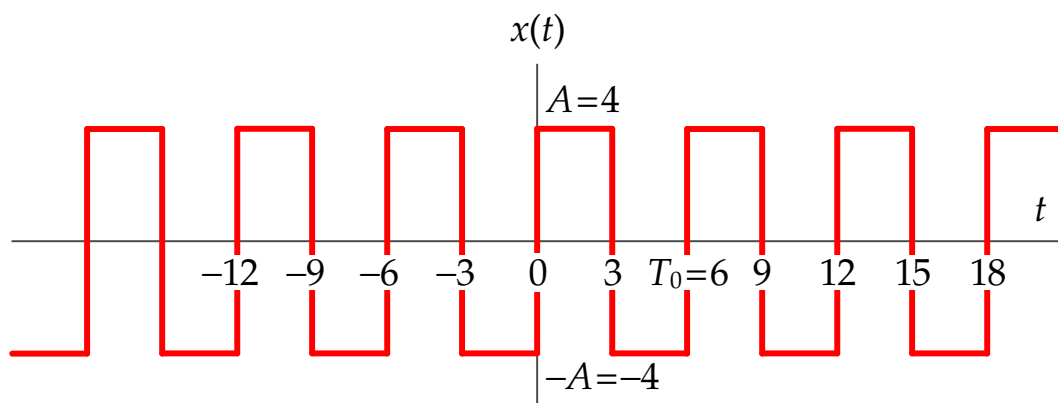
The average value is $\alpha_0 = \text{sinc} \left(\frac{0}{2} \right) = 1$.

$$c_n = 2|\alpha_n|$$

$$\theta_n = -\angle\alpha_n = \tan^{-1} \left(\frac{-\text{Im}\{\alpha_n\}}{\text{Re}\{\alpha_n\}} \right)$$

$\alpha_0 = \text{sinc}\left(\frac{0}{2}\right) = 1$	$ \alpha_0 = 1$	$\angle\alpha_0 = 0$
$\alpha_1 = \text{sinc}\left(\frac{1}{2}\right) = \frac{\sin(\pi/2)}{\pi/2} = \frac{+1}{\pi/2}$	$ \alpha_1 = \frac{2}{\pi} = 0.6366$	$\angle\alpha_1 = 0$
$\alpha_2 = \text{sinc}\left(\frac{2}{2}\right) = \frac{\sin(\pi)}{\pi} = \frac{0}{\pi}$	$ \alpha_2 = 0$	$\angle\alpha_2 = 0$
$\alpha_3 = \text{sinc}\left(\frac{3}{2}\right) = \frac{\sin(3\pi/2)}{3\pi/2} = \frac{-1}{3\pi/2}$	$ \alpha_3 = \frac{2}{3\pi} = 0.2122$	$\angle\alpha_3 = \pm\pi$
$\alpha_4 = \text{sinc}\left(\frac{4}{2}\right) = \frac{\sin(2\pi)}{2\pi} = \frac{0}{2\pi}$	$ \alpha_4 = 0$	$\angle\alpha_4 = 0$
$\alpha_5 = \text{sinc}\left(\frac{5}{2}\right) = \frac{\sin(5\pi/2)}{5\pi/2} = \frac{+1}{5\pi/2}$	$ \alpha_5 = \frac{2}{5\pi} = 0.1273$	$\angle\alpha_5 = 0$
....		

Q3. For the following signal $x(t) = \text{rep}_6 \left\{ 8 \text{rect}\left(\frac{t-1.5}{3}\right) \right\} - 4$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q3. Solution. The trigonometric Fourier series is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] , \quad \omega_0 = \frac{2\pi}{T_0}$$

where the coefficients are calculated as

$$a_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(n\omega_0 t) dt , \quad n = 0, 1, 2, 3, \dots$$

$$b_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(n\omega_0 t) dt , \quad n = 0, 1, 2, 3, \dots$$

Since $x(t)$ is odd, then $a_n = 0, \forall n$, and $b_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \sin(n\omega_0 t) dt$

$$b_n = \frac{4}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) \sin(n\omega_0 t) dt = \frac{2A}{n\pi} [1 - \cos(n\pi)], \quad \forall n$$

$$b_n = 0, \quad \forall n = \text{even}$$

$$b_n = \frac{4A}{n\pi} = \frac{16}{n\pi}, \quad \forall n = \text{odd}$$

$$a_n = 0, \quad \forall n$$

The compact Fourier series is

$$c_n = 2|\alpha_n| = \sqrt{a_n^2 + b_n^2} = |b_n|$$

$$\theta_n = -\angle\alpha_n = \tan^{-1} \left(\frac{-\text{Im}\{\alpha_n\}}{\text{Re}\{\alpha_n\}} \right) = \tan^{-1} \left(\frac{b_n}{a_n} \right) = \tan^{-1} \left(\frac{b_n}{0} \right) = \frac{\pi}{2}$$

$$\alpha_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt = 0$$

$$\alpha_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt = 0 - j \frac{1}{2} \times \frac{8}{n\pi} [1 - \cos(n\pi)] = \frac{a_n}{2} - j \frac{b_n}{2}$$

$$a_n = 0, \quad \forall n$$

$$b_n = \frac{2A}{n\pi} [1 - \cos(n\pi)] = \frac{8}{n\pi} [1 - \cos(n\pi)], \quad \forall n$$

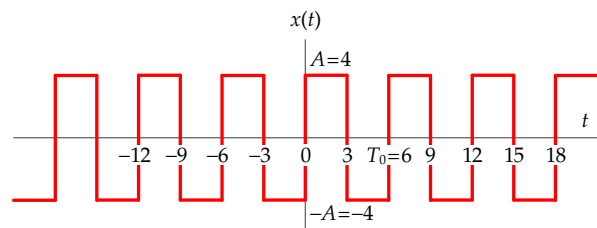
$$b_n = 0, \quad \forall n = \text{even}$$

$$b_n = \frac{4A}{n\pi} = \frac{16}{n\pi}, \quad \forall n = \text{odd}$$

Solutions below are equivalent:

$$\alpha_n = j \frac{A}{n\pi} [\cos(n\pi) - 1], \quad n \neq 0$$

$$\alpha_n = A \operatorname{sinc}\left(\frac{n}{2}\right) e^{-j\frac{n\pi}{2}}, \quad n \neq 0$$



But notice that we can use Fourier transform (and Fourier series) properties to quickly come up with the second equation,

$$\begin{aligned} \alpha_n &= \frac{2A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) e^{-jn\omega_0 t_0} = \frac{2AT_0/2}{T_0} \operatorname{sinc}\left(\frac{n\omega_0 T_0/2}{2\pi}\right) e^{-jn\omega_0 T_0/4} \\ &= A \operatorname{sinc}\left(\frac{n}{2}\right) e^{-j\frac{n\pi}{2}}, \quad n \neq 0 \end{aligned}$$

$$\alpha_n = A \operatorname{sinc}\left(\frac{n}{2}\right) e^{-j\frac{n\pi}{2}} = A \operatorname{sinc}\left(\frac{n}{2}\right) \left[\cos\left(\frac{n\pi}{2}\right) - j \sin\left(\frac{n\pi}{2}\right) \right]$$

$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = A \operatorname{sinc}\left(\frac{n}{2}\right) \cos\left(\frac{n\pi}{2}\right) - j A \operatorname{sinc}\left(\frac{n}{2}\right) \sin\left(\frac{n\pi}{2}\right)$$

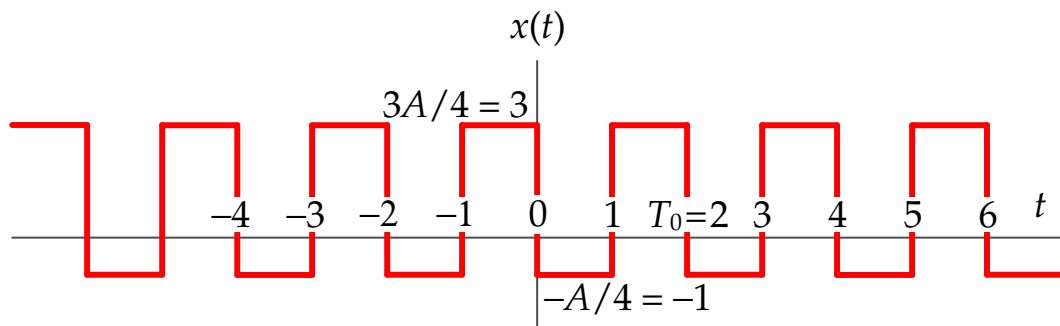
$$\alpha_n = \frac{a_n}{2} - j \frac{b_n}{2} = \frac{A \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} - j \frac{A \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}$$

$$a_n = \frac{4A \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{2}\right)}{n\pi} = 0, \quad \forall n$$

$$b_n = \frac{4A \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)}{n\pi} = 0, \quad \forall n = \text{even}$$

$$b_n = \frac{4A \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)}{n\pi} = \frac{4A}{n\pi}, \quad \forall n = \text{odd}$$

Q4. For the following signal $x(t) = \operatorname{rep}_2\{4 \operatorname{rect}(t + 0.5)\} - 1 = \operatorname{rep}_2\{-4 \operatorname{rect}(t - 0.5)\} + 3$, determine the complex exponential Fourier series, the trigonometric Fourier series and the compact Fourier series.



Q4. Solution.

$$\begin{aligned}\alpha_0 &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt \\ &= \frac{1}{T_0} \int_{-T_0/2}^0 (3A/4)(e^0) dt + \frac{1}{T_0} \int_0^{T_0/2} (-A/4)(e^0) dt \\ &= \frac{3A/4}{T_0} [t]_{-T_0/2}^0 + \frac{-A/4}{T_0} [t]_0^{T_0/2} = \frac{3A/4}{T_0} \left[0 - \frac{-T_0}{2} \right] + \frac{-A/4}{T_0} \left[\frac{T_0}{2} - 0 \right] \\ \alpha_0 &= \frac{3A}{8} + \frac{-A}{8} = \frac{2A}{8} = \frac{A}{4} = \frac{4}{4} = 1\end{aligned}$$

Now find α_n , a_n , b_n , c_n , and θ_n .

Using Fourier transform (and Fourier series) properties,

$$\begin{aligned}\alpha_n &= \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) e^{jn\omega_0 t_0} = \frac{AT_0/2}{T_0} \operatorname{sinc}\left(\frac{n\omega_0 T_0/2}{2\pi}\right) e^{jn\omega_0 T_0/4} \\ &= \frac{A}{2} \operatorname{sinc}\left(\frac{n}{2}\right) e^{+j\frac{n\pi}{2}}, \quad n \neq 0\end{aligned}$$

Only α_0 is affected by the amplitude shifting (DC shift)

$$\alpha_0 = \frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) e^0 - \frac{A}{4} = \frac{AT_0/2}{T_0} \operatorname{sinc}(0) - \frac{A}{4} = \frac{A}{2} - \frac{A}{4} = \frac{A}{4} = \frac{4}{4} = 1$$

HW4. Solution using Integration.

$$\begin{aligned}\alpha_n &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jn\omega_0 t} dt \\ &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^0 \left(\frac{3A}{4}\right) (e^{-jn\omega_0 t}) dt + \frac{1}{T_0} \int_0^{\frac{T_0}{2}} \left(-\frac{A}{4}\right) (e^{-jn\omega_0 t}) dt, \quad n \neq 0 \\ \alpha_n &= \frac{3A}{4T_0} \left[\frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \right]_{-\frac{T_0}{2}}^0 + \frac{-A}{4T_0} \left[\frac{1}{-jn\omega_0} e^{-jn\omega_0 t} \right]_0^{\frac{T_0}{2}} \\ \alpha_n &= \frac{3A}{4T_0} \times \frac{1}{-jn\omega_0} \left[e^0 - e^{\frac{+jn\omega_0 T_0}{2}} \right] + \frac{-A}{4T_0} \times \frac{1}{-jn\omega_0} \left[e^{\frac{-jn\omega_0 T_0}{2}} - e^0 \right]\end{aligned}$$

$$\begin{aligned}\alpha_n &= \frac{3A}{4T_0} \times \frac{1}{-jn\omega_0} \left[e^0 - e^{\frac{+jn\omega_0 T_0}{2}} \right] + \frac{-A}{4T_0} \times \frac{1}{-jn\omega_0} \left[e^{\frac{-jn\omega_0 T_0}{2}} - e^0 \right] \\ \alpha_n &= \frac{3A}{4T_0} \times \frac{e^{\frac{+jn\omega_0 T_0}{4}}}{-jn\omega_0} \left[e^{\frac{-jn\omega_0 T_0}{4}} - e^{\frac{+jn\omega_0 T_0}{4}} \right] + \frac{-A}{4T_0} \times \frac{e^{\frac{-jn\omega_0 T_0}{4}}}{-jn\omega_0} \left[e^{\frac{-jn\omega_0 T_0}{4}} - e^{\frac{+jn\omega_0 T_0}{4}} \right] \\ \alpha_n &= \frac{3A}{2T_0} \times \frac{e^{\frac{+jn\omega_0 T_0}{4}}}{n\omega_0} \left[\frac{e^{\frac{+jn\omega_0 T_0}{4}} - e^{\frac{-jn\omega_0 T_0}{4}}}{2j} \right] + \frac{-A}{2T_0} \times \frac{e^{\frac{-jn\omega_0 T_0}{4}}}{n\omega_0} \left[\frac{e^{\frac{+jn\omega_0 T_0}{4}} - e^{\frac{-jn\omega_0 T_0}{4}}}{2j} \right] \\ \alpha_n &= \frac{3A}{2T_0} \times \frac{e^{\frac{+jn\omega_0 T_0}{4}}}{n\omega_0} \sin\left(\frac{n\omega_0 T_0}{4}\right) + \frac{-A}{2T_0} \times \frac{e^{\frac{-jn\omega_0 T_0}{4}}}{n\omega_0} \sin\left(\frac{n\omega_0 T_0}{4}\right)\end{aligned}$$

$$\alpha_n = \frac{3A}{2T_0} \times \frac{e^{\frac{+jn\omega_0 T_0}{4}}}{n\omega_0} \sin\left(\frac{n\omega_0 T_0}{4}\right) + \frac{-A}{2T_0} \times \frac{e^{\frac{-jn\omega_0 T_0}{4}}}{n\omega_0} \sin\left(\frac{n\omega_0 T_0}{4}\right)$$

$$\alpha_n = \frac{3A}{8} \times \frac{e^{\frac{+jn\omega_0 T_0}{4}}}{\pi \frac{n\omega_0 T_0}{4\pi}} \times \sin\left(\pi \frac{n\omega_0 T_0}{4\pi}\right) + \frac{-A}{8} \times \frac{e^{\frac{-jn\omega_0 T_0}{4}}}{\pi \frac{n\omega_0 T_0}{4\pi}} \times \sin\left(\pi \frac{n\omega_0 T_0}{4\pi}\right)$$

$$\alpha_n = \frac{3A}{8} \times e^{\frac{+jn\omega_0 T_0}{4}} \times \text{sinc}\left(\frac{n\omega_0 T_0}{4\pi}\right) + \frac{-A}{8} \times e^{\frac{-jn\omega_0 T_0}{4}} \times \text{sinc}\left(\frac{n\omega_0 T_0}{4\pi}\right)$$

Notice,

$$\frac{n\omega_0 T_0}{4\pi} = \frac{n \times 2\pi \times T_0}{4\pi \times T_0} = \frac{n}{2}$$

$$\frac{n\omega_0 T_0}{4} = \frac{n \times 2\pi \times T_0}{4 \times T_0} = \frac{n\pi}{2}$$

Hence,

$$\alpha_n = \frac{3A}{8} \times e^{+j\frac{n\pi}{2}} \times \text{sinc}\left(\frac{n}{2}\right) + \frac{-A}{8} \times e^{-j\frac{n\pi}{2}} \times \text{sinc}\left(\frac{n}{2}\right)$$

Notice for $\forall n = \text{even}$

$$\text{sinc}\left(\frac{n}{2}\right) = 0$$

$$\alpha_n = 0$$

which is equal in value to,

$$\alpha_n = \frac{A}{2} \text{sinc}\left(\frac{n}{2}\right) e^{-j\frac{n\pi}{2}}$$

For $\forall n = \text{odd}$, we notice from Euler's identity that,

$$\begin{aligned} e^{+j\frac{n\pi}{2}} &= \cos\left(\frac{n\pi}{2}\right) + j \sin\left(\frac{n\pi}{2}\right) = 0 + j \sin\left(\frac{n\pi}{2}\right) = -\left[0 - j \sin\left(\frac{n\pi}{2}\right)\right] \\ &= -\left[\cos\left(\frac{n\pi}{2}\right) - j \sin\left(\frac{n\pi}{2}\right)\right] = -e^{-j\frac{n\pi}{2}} \end{aligned}$$

In other words, two conjugate numbers are equal and negative of each other because the real part is zero $\forall n = \text{odd}$. Hence,

$$\begin{aligned} \alpha_n &= \frac{3A}{8} \times e^{+j\frac{n\pi}{2}} \times \text{sinc}\left(\frac{n}{2}\right) + \frac{-A}{8} \times e^{-j\frac{n\pi}{2}} \times \text{sinc}\left(\frac{n}{2}\right) \\ \alpha_n &= \frac{3A}{8} \times e^{+j\frac{n\pi}{2}} \times \text{sinc}\left(\frac{n}{2}\right) + \frac{A}{8} \times e^{+j\frac{n\pi}{2}} \times \text{sinc}\left(\frac{n}{2}\right) \\ \alpha_n &= \left[\frac{3A}{8} + \frac{A}{8}\right] \times e^{+j\frac{n\pi}{2}} \times \text{sinc}\left(\frac{n}{2}\right) = \frac{A}{2} \text{sinc}\left(\frac{n}{2}\right) e^{+j\frac{n\pi}{2}}, \quad n \neq 0 \end{aligned}$$